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SO($2n+1$) in an SO($2n-3$) $\otimes$ SU(2) $\otimes$ SU(2) basis: I. Reduction of the symmetric representations

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Abstract. The branching rule for the reduction of symmetric irreducible unitary representations (IUR) of the simple Lie group SO($2n+1$) into IUR of its maximal subgroup SO($2n-3$) $\otimes$ SU(2) $\otimes$ SU(2) is established for all $n \geq 3$. After the particular case $n = 3$ is analysed in detail, a general proof is presented which is valid for all $n \geq 3$. All branching rules ($n = 3, 4, \ldots$) can be summmed up in one formula. Also, a dimension verification is carried out. The generators of SO($2n+1$) not belonging to the semi-simple subgroup can be combined into a mixed tensor–spinor representation with respect to the simple groups which occur in the direct product. The precise nature of that representation is indicated and discussed.

1. Introduction

The problem of obtaining branching rules for representations of semi-simple Lie groups has been extensively investigated during the past decades. In many cases, however, such a branching is only explicitly established in the form of multiplicity tables, of which a wide variety is found in the standard literature (Wybourne 1970, McKay and Patera 1981).

A few years ago two of the present authors (G Vanden Berghe and H De Meyer) took up interest in the nuclear multipole phonon state labelling problem, which arises in the nuclear collective model. Due to the bosonic character of the phonons, the problem is associated to that of the complete classification of states belonging to symmetric irreducible representations of the rotation groups SO($2n+1$), $n = 2, 3, \ldots$. Also since SO(3) $\subset$ SO($2n+1$), it is usually demanded to construct orthonormal bases of SO($2n+1$) states which are angular momentum states too, in other terms, states for which angular momentum l and its projection m are good quantum numbers. Unfortunately, in the reduction of symmetric IUR of SO($2n+1$) into IUR of its principal SO(3) subgroup, in general one or more labels are missing. An interesting solution to the quadrupole ($n = 2$) phonon one missing label problem has been given by Kemmer *et al* (1968) and Williams and Pursey (1968). These authors first consider the reduction of SO(5) into the maximal subgroup SU(2) $\otimes$ SU(2). It turns out that in this reduction no degeneracies occur. The SU(2) $\otimes$ SU(2) states are then combined into SO(3) states by means of the Hill–Wheeler projection, a method adapted to the

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fact that the physical principal SO(3) subgroup of SO(5) is not contained in $SU(2) \otimes SU(2)$.

In order to extend this procedure to the octupole ($n = 3$) phonon case, and eventually to higher multipole phonons ($n > 3$) too, we treat in the present paper, as a first step, the reduction of the symmetric representations of $SO(2n+1)$ into IUR of its maximal subgroup $SO(2n-3) \otimes SU(2) \otimes SU(2)$ (Dynkin 1965b). In § 2 we derive the corresponding branching rule for $n = 3$ using a technique described by Stone and applied by him to second-rank group reductions (Stone 1970). Section 3 is concerned with the generalisation of the proof for all $n \geq 3$. In every case it is found that in the reduction of the symmetric representations degeneracies are absent. What is more, all branching rules can be summarised in one simple formula. For the sake of completeness a verification on the dimension of the representations is carried out. Also the reduction $SO(2n+1) \to [\otimes SU(2)]^n$ is discussed. In § 4 it is indicated how the generators of $SO(2n+1)$ which do not belong to the subgroup $SO(2n-3) \otimes SU(2) \otimes SU(2)$ can be combined into a mixed tensor operator which behaves as a vector with respect to $SO(2n-3)$ and as a spinor with respect to both SU(2).

2. Branching $SO(7) \to SU(2) \otimes SU(2) \otimes SO(3)$

In order to derive the branching rule for the symmetric IUR of SO(7) into IUR of $SU(2) \otimes SU(2) \otimes SO(3)$ we shall apply hereafter a method developed by Stone (1970). The proof is a purely algebraic one, and therefore we convert all notions concerning the branching of representations into Lie algebraic language. Hence, we need to consider in Cartan's notation the branching $B_3 \to A_1^1 \oplus A_1^1 \oplus A_1^2$, where B_3 is the Lie algebra of SO(7) and A_1 the Lie algebra of SU(2) or SO(3). Lower indices refer to the rank of the algebra, whereas upper indices denote the index of the subalgebra (Dynkin 1965a), which can be geometrically interpreted here as the square of the ratio of the length of the longest simple root of B_3 to the length of the simple root of A_1.

Usually, the nine positive roots of B_3—more accurately root forms—are described in an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ as follows:

$$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 \pm \boldsymbol{e}_2, \boldsymbol{e}_1 \pm \boldsymbol{e}_3, \boldsymbol{e}_2 \pm \boldsymbol{e}_3. \tag{1}$$

The three simple roots $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ are found to be

$$\boldsymbol{\alpha}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2, \qquad \boldsymbol{\alpha}_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3, \qquad \boldsymbol{\alpha}_3 = \boldsymbol{e}_3. \tag{2}$$

It has to be noticed that the numbering of these roots is not the one uniquely encountered, but this fact is irrelevant to our purpose. The irreducible representations of B_3 are labelled by their highest weight components with respect to the $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ basis, i.e. $[w_1, w_2, w_3]$ denotes the representation with highest weight

$$\boldsymbol{\Lambda} = w_1\boldsymbol{e}_1 + w_2\boldsymbol{e}_2 + w_3\boldsymbol{e}_3 = w_1\boldsymbol{\alpha}_1 + (w_1 + w_2)\boldsymbol{\alpha}_2 + (w_1 + w_2 + w_3)\boldsymbol{\alpha}_3. \tag{3}$$

Also $w_1 \geq w_2 \geq w_3 \geq 0, \frac{1}{2}$. Further on, we shall consider only the symmetric representations $[v, 0, 0]$ where v can be any non-negative integer.

In the root system of the algebra B_3 the appointment of three mutually orthogonal simple roots $\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2$, and $\boldsymbol{\alpha}'_3$ for the subalgebra $A_1^1 \oplus A_1^1 \oplus A_1^2$ is not unique. Indeed, from (1) it is clear that having defined $\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2$ and $\boldsymbol{\alpha}'_3$ in terms of $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$, any formal permutation of the latter basis vectors leads to another acceptable definition of the simple roots. However, since the branching rule should be independent of any particular choice, it is convenient to define

$$\boldsymbol{\alpha}'_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2, \qquad \boldsymbol{\alpha}'_2 = \boldsymbol{e}_1 - \boldsymbol{e}_2, \qquad \boldsymbol{\alpha}'_3 = \boldsymbol{e}_3. \tag{4}$$

Notice that in Dynkin's terminology (Dynkin 1965b) the subalgebra is also regular. The irreducible representations of $A_1^1 \oplus A_1^1 \oplus A_1^2$ are labelled by a triple $(s, t, u)'$ of A_1-representation labels s, t, u which can take integer or half odd-integer values. Also s, t and u, in this order, form the components in the $\{\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, \boldsymbol{\alpha}'_3\}$ basis of the highest weight Λ' contained in the subalgebra representation.

At this point it should be clear that we shall constantly use primes when we refer to the subalgebra.

We next introduce certain concepts which are of fundamental importance here. A particular root form, namely half the sum of the positive roots of an algebra, for which we reserve the symbol $\boldsymbol{R}$, is one such concept. We easily find

$$\boldsymbol{R} = \tfrac{1}{2}(5\boldsymbol{e}_1 + 3\boldsymbol{e}_2 + \boldsymbol{e}_3) = \tfrac{1}{2}(5\boldsymbol{\alpha}_1 + 8\boldsymbol{\alpha}_2 + 9\boldsymbol{\alpha}_3), \tag{5a}$$

$$\boldsymbol{R}' = \tfrac{1}{2}(2\boldsymbol{e}_1 + \boldsymbol{e}_3) = \tfrac{1}{2}(2\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + 3\boldsymbol{\alpha}_3). \tag{5b}$$

Another concept which we need is that of the Weyl reflection group (Wybourne 1974). This is the finite group generated by all the reflections S_{α_i} (with $\boldsymbol{\alpha}_i$ any simple root),

$$S_{\alpha_i}: \boldsymbol{K} \to \boldsymbol{K}_{\alpha_i} = \boldsymbol{K} - 2\frac{\boldsymbol{K} \cdot \boldsymbol{\alpha}_i}{\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_i} \cdot \boldsymbol{\alpha}_i, \tag{6}$$

which in weight space transform a weight $\boldsymbol{K}$ into $\boldsymbol{K}_{\alpha_i}$ which is the reflected weight with respect to the hyperplane normal to $\boldsymbol{\alpha}_i$. Let us construct now the set of all weights which under the SO(7) Weyl reflection group are equivalent to a particular weight $\boldsymbol{K} = (k_1, k_2, k_3)_e = k_1\boldsymbol{e}_1 + k_2\boldsymbol{e}_2 + k_3\boldsymbol{e}_3$. To that aim we first deduce with the help of the definitions (2) and (6) that

$$\begin{aligned} &S_{\alpha_1}(k_1, k_2, k_3)_e = (k_2, k_1, k_3)_e, \\ &S_{\alpha_2}(k_1, k_2, k_3)_e = (k_1, k_3, k_2)_e, \\ &S_{\alpha_3}(k_1, k_2, k_3)_e = (k_1, k_2, -k_3)_e. \end{aligned} \tag{7}$$

Hence, the set of equivalent weights consists of the 48 elements of the form

$$\{S(\boldsymbol{K})\} = \{(\varepsilon_1 k_{\pi(1)}, \varepsilon_2 k_{\pi(2)}, \varepsilon_3 k_{\pi(3)})_e\}, \tag{8}$$

where π is a permutation of $(1, 2, 3)$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+1, -1\}$. To each weight (8) is associated a parity factor δ_S which equals $+1$ if an even number of basic reflections (7) is needed to obtain (8), and -1 if that number is odd.

Finally we introduce the following expressions:

$$\xi(\boldsymbol{K}) = \sum_S \delta_S \exp[S(\boldsymbol{K} + \boldsymbol{R})] \tag{9}$$

$$\Delta = \exp(\boldsymbol{R}) \prod_{\boldsymbol{\alpha}^+} (1 - \exp(-\boldsymbol{\alpha}^+)), \tag{10}$$

$$\Delta' = \exp(\boldsymbol{R}') \prod_{\boldsymbol{\alpha}'^+} (1 - \exp(-\boldsymbol{\alpha}'^+)), \tag{11}$$

where the products in (10) and (11) extend over the positive roots of the algebra and subalgebra respectively. We are now in a position to apply immediately a formula of

Stone (1970) which produces the multiplicity $n(s, t, u)$, which is the number of times the $A_1^1 \oplus A_1^1 \oplus A_1^2$ representation $(s, t, u)'$ with highest weight $\boldsymbol{\Lambda}' = s\boldsymbol{\alpha}_1' + t\boldsymbol{\alpha}_2' + u\boldsymbol{\alpha}_3'$ occurs in the reduction of the B_3 representation $[w_1, w_2, w_3]$ with highest weight $\boldsymbol{\Lambda} = w_1\boldsymbol{e}_1 + w_2\boldsymbol{e}_2 + w_3\boldsymbol{e}_3$. More precisely, we learn from Stone's paper that $n(s, t, u)$ is the constant term in the formal power series development of the expression

$$\exp(-\boldsymbol{\Lambda}' - \boldsymbol{R}')\xi(\boldsymbol{\Lambda})\Delta'/\Delta, \tag{12}$$

in powers of $\exp(-\boldsymbol{\alpha}_i)$ $(i = 1, 2, 3)$. If we substitute in formula (12) the expressions (9), (10) and (11), also taking into account the definitions (5), if we take care that every vector is expressed in terms of the simple B_3 roots $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$, and if we then introduce the formal notations $x = \exp(-\boldsymbol{\alpha}_1)$, $y = \exp(-\boldsymbol{\alpha}_2)$, $z = \exp(-\boldsymbol{\alpha}_3)$, we straightforwardly deduce from (8) that $n(s, t, u)$ is the constant term in the expansion of the expression

$$\sum_S \delta_S x^{s+t+5/2-\varepsilon_1 k_{\pi(1)}} y^{2s+4-\varepsilon_1 k_{\pi(1)}-\varepsilon_2 k_{\pi(2)}} z^{2s+u+9/2-\varepsilon_1 k_{\pi(1)}-\varepsilon_2 k_{\pi(2)}-\varepsilon_3 k_{\pi(3)}}$$
$$\times (1-y)^{-1}(1-yz)^{-1}(1-yz^2)^{-1}(1-xy)^{-1}(1-xyz)^{-1}(1-xyz^2)^{-1}, \tag{13}$$

in powers of x, y and z, and in which

$$k_1 = w_1 + 5/2, \qquad k_2 = w_2 + 3/2, \qquad k_3 = w_3 + 1/2. \tag{14}$$

In general, the 48 terms of the sum in (13) can contribute to the constant term of the expansion, and this fact excludes at first sight a full analysis. However, if we restrict to symmetric representations $[v, 0, 0]$, with the consequence that $k_1 = v + 5/2$, $k_2 = 3/2$ and $k_3 = 1/2$, only eight terms of the sum must be taken into consideration. Indeed, in order that such a term contributes to the constant term of the series development, it is a necessary condition that the exponents of the accompanying powers of x, y and z are non-positive integers. The only way to satisfy this condition for the exponent of x, is to require that $\varepsilon_1 k_{\pi(1)} = v + 5/2$, or equivalently that the permutation π is such that $\pi(1) = 1$, and $\varepsilon_1 = +1$. Since this restricts the number of possible permutations π to two, and ε_2 and ε_3 can be independently attributed the values $+1$ and -1, only eight terms in (13) survive. If, furthermore, we pass in (13) to three new variables $x' = xy$, $y' = y$, $z' = z$, $n(s, t, u)$ becomes the constant term in the expansion of

$$\sum_{S'} \delta_{S'} x'^{s+t-v} y'^{s-t+3/2-\varepsilon_2 k_{\pi(2)}} z'^{2s+u-v+2-\varepsilon_2 k_{\pi(2)}-\varepsilon_3 k_{\pi(3)}}$$
$$\times (1-x')^{-1}(1-x'z')^{-1}(1-x'z'^2)^{-1}(1-y')^{-1}(1-y'z')^{-1}(1-y'z'^2)^{-1}. \tag{15}$$

Then, it is straightforward to expand the six reciprocals contained in (15), and to deduce thereafter that

$$\begin{aligned} n(s, t, u) = \sum_{\lambda,\mu,\nu,\rho,\sigma,\tau \geqslant 0} \delta_{v-s-t,\lambda+\mu+\nu} \{ & \delta_{t-s,\rho+\sigma+\tau}\delta_{v-2s-u,\mu+2\nu+\sigma+2\tau} \\ & + \delta_{t-s-1,\rho+\sigma+\tau}\delta_{v-2s-u-3,\mu+2\nu+\sigma+2\tau} + \delta_{t-s-2,\rho+\sigma+\tau}\delta_{v-2s-u-1,\mu+2\nu+\sigma+2\tau} \\ & + \delta_{t-s-3,\rho+\sigma+\tau}\delta_{v-2s-u-4,\mu+2\nu+\sigma+2\tau} - \delta_{t-s,\rho+\sigma+\tau}\delta_{v-2s-u-1,\mu+2\nu+\sigma+2\tau} \\ & - \delta_{t-s-1,\rho+\sigma+\tau}\delta_{v-2s-u,\mu+2\nu+\sigma+2\tau} - \delta_{t-s-2,\rho+\sigma+\tau}\delta_{v-2s-u-4,\mu+2\nu+\sigma+2\tau} \\ & - \delta_{t-s-3,\rho+\sigma+\tau}\delta_{v-2s-u-3,\mu+2\nu+\sigma+2\tau} \} \end{aligned} \tag{16}$$

where the δ's denote Kronecker deltas. The eight δ-product terms on the RHS of (16) correspond to the eight separate terms in the S' summation in (15). In order to reduce the RHS of (16) we can proceed as follows.

In the sixfold sum over δ-products of the first type we separate the $\rho=0$ contributions, then we replace ρ by $\rho'+1$, where ρ' runs from zero to infinity, and finally we omit primes and make use of the property that $\delta_{a,b+1}=\delta_{a-1,b}$. But, in this way, we exactly obtain the δ-products of the sixth type, and since the signs in front are opposite, both cancel each other. Hence, from the first and sixth type of δ-products only the $\rho=0$ contributions of the first type remain. Similarly, from the second and fifth type only the $\tau=0$ contributions of the fifth type remain, from the third and eighth the $\tau=0$ contributions of the third, and from the fourth and seventh the $\rho=0$ contributions of the seventh. Finally, replacing the summation index τ by ρ, we obtain

$$\begin{aligned} n(s,t,u)=\sum_{\lambda,\mu,\nu,\rho,\sigma\geq 0} \delta_{v-s-t,\lambda+\mu+\nu}\{&\delta_{t-s,\rho+\sigma}\delta_{v-2s-u,\mu+2\nu+\sigma+2\rho} \\ &+\delta_{t-s-2,\rho+\sigma}\delta_{v-2s-u-1,\mu+2\nu+\sigma}-\delta_{t-s,\rho+\sigma}\delta_{v-2s-u-1,\mu+2\nu+\sigma} \\ &-\delta_{t-s-2,\rho+\sigma}\delta_{v-2s-u-4,\mu+2\nu+\sigma+2\rho}\}. \end{aligned} \tag{17}$$

The process of reduction can be repeated on the δ-product sums on the RHS of (17). The ρ summation can be cancelled by separating the $\rho=0$ and $\rho=1$ contributions from the first and third type of products. We leave it to the reader that similar operations can be carried out until only two summation indices are left. At that point we obtain

$$n(s,t,u)=\sum_{\mu,\nu\geq 0}\{\delta_{v-s-t,\mu+\nu}\delta_{t-s,0}\delta_{v-2s-u,\nu}-\delta_{v-s-t,\mu+\nu}\delta_{t-s,0}\delta_{v-2s-u-1,2\nu}\} \tag{18}$$

$$=\delta_{t,s}\sum_{\mu,\nu\geq 0}\delta_{v-2s,\mu+\nu}\delta_{v-2s-u,2\nu}. \tag{19}$$

Hence

$$n(s,t,u)=\begin{cases}+1 & \text{if } s=t\leq v/2 \text{ and } v-2s-u=\text{even},\\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

Consequently, no degeneracies occur in the reduction of the symmetric B_3 representations, and the branching rule for that class of representations can be formulated as follows:

$$[v,0,0]\to\sum_{\mu=0}^{v}\sum_{\nu=0}^{[(v-\mu)/2]}(\mu/2,\mu/2,v-\mu-2\nu)', \tag{21}$$

where $[r]$ denotes the largest integer not larger than r.

3. Branching SO(2n + 1)→SU(2) ⊗ SU(2) ⊗ SO(2n − 3)

In the present section we want to generalise the results to the reduction of symmetric representations of B_n into representations of the subalgebra $A_1^1\oplus A_1^1\oplus B_{n-2}$ for $n>3$. In an orthonormal basis $\{e_i|i=1,\ldots,n\}$ the positive roots of B_3 are

$$e_i\quad 1\leq i\leq n,\qquad e_i\pm e_j\quad 1\leq i<j\leq n. \tag{22}$$

In terms of the simple roots $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$, the positive roots (22) can be re-expressed as

$$\begin{aligned} &\boldsymbol{e}_i = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_{i+1} + \ldots + \boldsymbol{\alpha}_n, \qquad 1 \leqslant i \leqslant n, \\ &\boldsymbol{e}_i + \boldsymbol{e}_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} + 2\boldsymbol{\alpha}_j + 2\boldsymbol{\alpha}_{j+1} + \ldots + 2\boldsymbol{\alpha}_n \\ &\boldsymbol{e}_i - \boldsymbol{e}_j = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_{i+1} + \ldots + \boldsymbol{\alpha}_{j-1}, \qquad 1 \leqslant i < j \leqslant n. \end{aligned} \tag{23}$$

Next, we must define the root system of the subalgebra. A convenient choice consists in identifying the simple roots $\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, \ldots, \boldsymbol{\alpha}'_{n-2}$ of B_{n-2} with $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{n-2}$. Then according to the index of both A_1 algebras we assign to the first of these the simple root $\boldsymbol{\alpha}'_{n-1} = \boldsymbol{e}_{n-1} + \boldsymbol{e}_n = \boldsymbol{\alpha}_{n-1} + 2\boldsymbol{\alpha}_n$ and to the second the simple root $\boldsymbol{\alpha}'_n = \boldsymbol{e}_{n-1} - \boldsymbol{e}_n = \boldsymbol{\alpha}_{n-1}$. In order to apply Stone's formula we need to know which positive roots of B_3 are not a root of the subalgebra $\mathrm{A}_1 \oplus \mathrm{A}_1 \oplus \mathrm{B}_{n-2}$. Clearly these are

$$\begin{aligned} &\boldsymbol{e}_{n-1} = \boldsymbol{\alpha}_{n-1} + \boldsymbol{\alpha}_n, \qquad \boldsymbol{e}_n = \boldsymbol{\alpha}_n, \\ &\boldsymbol{e}_i + \boldsymbol{e}_{n-1} = \boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-2} + 2\boldsymbol{\alpha}_{n-1} + 2\boldsymbol{\alpha}_n, \qquad 1 \leqslant i \leqslant n-2, \\ &\boldsymbol{e}_i + \boldsymbol{e}_n = \boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-2} + \boldsymbol{\alpha}_{n-1} + 2\boldsymbol{\alpha}_n, \qquad 1 \leqslant i \leqslant n-2, \\ &\boldsymbol{e}_i - \boldsymbol{e}_{n-1} = \boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-2}, \qquad 1 \leqslant i \leqslant n-2, \\ &\boldsymbol{e}_i - \boldsymbol{e}_n = \boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-1}, \qquad 1 \leqslant i \leqslant n-2. \end{aligned} \tag{24}$$

For further calculations, it is opportune to introduce n new vectors $\boldsymbol{\beta}_i$ ($i = 1, \ldots, n$) defined as follows:

$$\begin{aligned} &\boldsymbol{\beta}_i = \boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-2}, \qquad 1 \leqslant i \leqslant n-2, \\ &\boldsymbol{\beta}_{n-1} = \boldsymbol{\alpha}_{n-1}, \qquad \boldsymbol{\beta}_n = \boldsymbol{\alpha}_n. \end{aligned} \tag{25}$$

Consequently, the positive roots of B_3 which are not a root of the subalgebra are

$$\begin{aligned} &\boldsymbol{\beta}_n, \boldsymbol{\beta}_{n-1} + \boldsymbol{\beta}_n, \boldsymbol{\beta}_i, \boldsymbol{\beta}_i + \boldsymbol{\beta}_{n-1}, \boldsymbol{\beta}_i + \boldsymbol{\beta}_{n-1} + 2\boldsymbol{\beta}_n, \boldsymbol{\beta}_i + 2\boldsymbol{\beta}_{n-1} + 2\boldsymbol{\beta}_n \\ &(i = 1, \ldots, n-2). \end{aligned} \tag{26}$$

It is also straightforward to deduce that

$$\begin{aligned} \boldsymbol{R} &= \tfrac{1}{2} \sum_{i=1}^{n} (2n+1-2i)\boldsymbol{e}_i = \tfrac{1}{2} \sum_{i=1}^{n} i(2n-i)\boldsymbol{\alpha}_i \\ &= \tfrac{1}{2} \sum_{i=1}^{n-2} (2n+1-2i)\boldsymbol{\beta}_i + \tfrac{1}{2}(n^2-1)\boldsymbol{\beta}_{n-1} + \tfrac{1}{2}n^2\boldsymbol{\beta}_n, \end{aligned} \tag{27a}$$

$$\begin{aligned} \boldsymbol{R}' &= \tfrac{1}{2} \sum_{i=1}^{n-2} (2n-3-2i)\boldsymbol{e}_i + \boldsymbol{e}_{n-1} \\ &= \tfrac{1}{2} \sum_{i=1}^{n-2} i(2n-i-4)\boldsymbol{\alpha}_i + \tfrac{1}{2}(n^2-4n+6)(\boldsymbol{\alpha}_{n-1} + \boldsymbol{\alpha}_n) \\ &= \tfrac{1}{2} \sum_{i=1}^{n-2} (2n-3-2i)\boldsymbol{\beta}_i + \tfrac{1}{2}(n^2-4n+6)(\boldsymbol{\beta}_{n-1} + \boldsymbol{\beta}_n). \end{aligned} \tag{27b}$$

Furthermore, if we define $x_i = \exp(-\boldsymbol{\beta}_i)$ $(i = 1, \ldots, n)$, it is easily seen on account of (26) and (27) that

$$\frac{\Delta'}{\Delta} = \left(\prod_{i=1}^{n-2} x_i^2\right) x_{n-1}^{2n-7/2} x_n^{2n-3} \left(\prod_{i=1}^{n-2} (1-x_i)^{-1}(1-x_i x_{n-1})^{-1}(1-x_i x_{n-1} x_n^2)^{-1} \right.$$
$$\left. \times (1-x_i x_{n-1}^2 x_n^2)^{-1}\right)(1-x_{n-1}x_n)^{-1}(1-x_n)^{-1}. \tag{28}$$

The symmetric representations of B_n are $[v, 0, \ldots, 0]$ with $n-1$ zeros. The representations of B_{n-2} will be labelled as $[u_1, u_2, \ldots, u_{n-2}]$ and those of both A_1's by s and t respectively. Hence, the $\mathrm{A}_1 \oplus \mathrm{A}_1 \oplus \mathrm{B}_{n-2}$ representations are labelled by $(s, t, [u_1, u_2, \ldots, u_{n-2}])'$, whereas the highest weight Λ' is given by

$$\begin{aligned}\Lambda' &= \sum_{i=1}^{n-2} u_i \boldsymbol{e}_i + (s+t)\boldsymbol{e}_{n-1} + (s-t)\boldsymbol{e}_n \\ &= \sum_{i=1}^{n-2} u_i \boldsymbol{\beta}_i + \left[\left(\sum_{i=1}^{n-2} u_i\right) + s + t\right]\boldsymbol{\beta}_{n-1} + \left[\left(\sum_{i=1}^{n-2} u_i\right) + 2s\right]\boldsymbol{\beta}_n.\end{aligned} \tag{29}$$

Consequently

$$\exp(-\Lambda' - \boldsymbol{R}') = \left(\prod_{i=1}^{n-2} x_i^{[u_i + (2n-3-2i)/2]}\right) x_{n-1}^{[\Sigma u_i + s + t + (n^2-4n+6)/2]} x_n^{[\Sigma u_i + 2s + (n^2-4n+6)/2]}. \tag{30}$$

Finally, since from (23) we learn that $\boldsymbol{\alpha}_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ $(i \leqslant n-1)$ and $\boldsymbol{\alpha}_n = \boldsymbol{e}_n$, the set of Weyl reflected weights associated to the weight $\Lambda + \boldsymbol{R}$ consists of the $n!2^n$ elements, which in the $\{e_i\}$ basis have components which differ from those of $\Lambda + \boldsymbol{R} = (v + (2n-1)/2, (2n-3)/2, \ldots, 3/2, 1/2)_e$, by a permutation of the components and by a possible change of sign of any one of these. More explicitly:

$$\begin{aligned}\{S(\Lambda + \boldsymbol{R})\} &= \{(\varepsilon_1 k_{\pi(1)}, \ldots, \varepsilon_{n-1} k_{\pi(n-1)}, \varepsilon_n k_{\pi(n)})_e\} \\ &= \left\{\left(\varepsilon_1 k_{\pi(1)}, \ldots, \varepsilon_{n-2} k_{\pi(n-2)}, \sum_{i=1}^{n-1} \varepsilon_i k_{\pi(i)}, \sum_{i=1}^{n} \varepsilon_i k_{\pi(i)}\right)_\beta\right\}\end{aligned} \tag{31}$$

with π a permutation of $(1, 2, \ldots, n)$ and $\varepsilon_i \in \{-1, +1\}$. Also

$$k_i = v\delta_{1i} + n - i + 1/2 \qquad (i = 1, \ldots, n). \tag{32}$$

At this point we can again apply Stone's formula. The substitution of equations (28), (30) and (31) into the expression (12), together with the definition (9), yield that $n(s, t, [u_1, \ldots, u_{n-2}])$, the number of times the representation $(s, t, [u_1, \ldots, u_{n-2}])'$ occurs in the reduction of $[v, 0, \ldots, 0]$, is the constant term in the power series development of

$$\sum_S \delta_S \left(\prod_{i=1}^{n-2} x_i^{u_i + n - i + 1/2 - \varepsilon_i k_{\pi(i)}}\right) x_{n-1}^{\Sigma_{i=1}^{n-2} u_i + s + t + (n^2-1)/2 - \Sigma_{i=1}^{n-1} \varepsilon_i k_{\pi(i)}}$$
$$\times x_n^{\Sigma_{i=1}^{n-2} u_i + 2s + n^2/2 - \Sigma_{i=1}^{n} \varepsilon_i k_{\pi(i)}} \left(\prod_{i=1}^{n-2} (1-x_i)^{-1}(1-x_i x_{n-1})^{-1}\right.$$
$$\left.\times (1-x_i x_{n-1} x_n^2)^{-1}(1-x_i x_{n-1}^2 x_n^2)^{-1}\right)(1-x_{n-1}x_n)^{-1}(1-x_n)^{-1}. \tag{33}$$

Again, it is clear that the powers of x_i $(i = 1, \ldots, n)$ in front of the reciprocal factors in (33) should have non-positive exponents in order to contribute to the constant term of the development. On account of (32) this can only be achieved for the exponents of x_i $(i=1,\ldots,n-2)$ if $\varepsilon_i k_{\pi(i)}=k_i$ $(i=1,\ldots,n-2)$. But, even then we notice that $u_2, u_3, \ldots, u_{n-2}$ are necessarily zero. Hence, only symmetric B_{n-2} representations occur in the reduction. Let us also omit the index 1 of u_1 and denote the B_{n-2} representations shortly as $[u]$. Then (33) can be considerably simplified. More precisely, $n(s,t,[u])$ is the constant term in the development of

$$\begin{aligned}\sum_{S'} \delta_{S'} x_1^{u-v} x_{n-1}^{u+s+t+3/2-v-\varepsilon_{n-1}k_{\pi(n-1)}} x_n^{u+2s+2-v-\varepsilon_{n-1}k_{\pi(n-1)}-\varepsilon_n k_{\pi(n)}}&\\ \times(1-x_1)^{-1}(1-x_1x_{n-1})^{-1}(1-x_1x_{n-1}x_n^2)^{-1}&\\ \times(1-x_1x_{n-1}^2x_n^2)^{-1}(1-x_{n-1}x_n)^{-1}(1-x_n)^{-1},&\qquad(34)\end{aligned}$$

where S' refers to the two possibilities $k_{\pi(n-1)}=3/2$, $k_{\pi(n)}=1/2$ and $k_{\pi(n-1)}=1/2$, $k_{\pi(n)}=3/2$ and the four independent ways to attribute to ε_{n-1} and ε_n the values $+1$ or -1. Making now the required expansions in (34), we arrive at

$$\begin{aligned}n(s,t,[u])=\sum_{\lambda,\mu,\nu,\rho,\sigma,\tau\geqslant 0}&\delta_{v-u,\lambda+\mu+\nu+\rho}\\ &\times\{\delta_{v-u-s-t,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s,2\nu+2\rho+\sigma+\tau}\\ &+\delta_{v-u-s-t-1,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-3,2\nu+2\rho+\sigma+\tau}\\ &+\delta_{v-u-s-t-2,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-1,2\nu+2\rho+\sigma+\tau}\\ &+\delta_{v-u-s-t-3,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-4,2\nu+2\rho+\sigma+\tau}\\ &-\delta_{v-u-s-t,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-1,2\nu+2\rho+\sigma+\tau}\\ &-\delta_{v-u-s-t-1,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s,2\nu+2\rho+\sigma+\tau}\\ &-\delta_{v-u-s-t-2,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-4,2\nu+2\rho+\sigma+\tau}\\ &-\delta_{v-u-s-t-3,\mu+\nu+2\rho+\sigma}\delta_{v-u-2s-3,2\nu+2\rho+\sigma+\tau}\}.\qquad(35)\end{aligned}$$

The reduction of the RHS of (35) proceeds in the same way as in § 2. To begin with, one separates the $\tau=0$ contributions out of the first delta product type. What is left exactly cancels the contributions of the fifth type. Similar cancellations occur if one separates $\tau=0$ from the eighth type, $\sigma=0$ from the second and $\sigma=0$ from the sixth. We leave it again to the reader to verify that one can achieve the formula

$$\begin{aligned}n(s,t,[u])=\sum_{\lambda,\mu\geqslant 0}&\delta_{v-u,\lambda+\mu}\delta_{v-u-2s,2\lambda}[\delta_{v-u-s-t,2\lambda}-\delta_{v-u-s-t-1,\lambda+\mu}]\\ &+\sum_{\lambda\geqslant 0}\delta_{v-u,\lambda}\delta_{v-u-2s-2,2\lambda}[\delta_{v-u-s-t-2,2\lambda}-\delta_{v-u-s-t-1,\lambda}].\qquad(36)\end{aligned}$$

In the double sum over λ and μ the contribution is zero unless $v-u=\lambda+\mu$. Also the product $\delta_{v-u,\lambda+\mu}\delta_{v-u-s-t-1,\lambda+\mu}$ is always zero since $s+t+1\neq 0$. In the same way one proves that the second sum over λ completely vanishes. Consequently

$$\begin{aligned}n(s,t,[u])&=\sum_{\lambda,\mu\geqslant 0}\delta_{v-u,\lambda+\mu}\delta_{v-u-2s,2\lambda}\delta_{v-u-s-t,2\lambda}\\ &=\delta_{s,t}\sum_{\lambda,\mu\geqslant 0}\delta_{v-u,\lambda+\mu}\delta_{v-u-2s,2\lambda}.\qquad(37)\end{aligned}$$

Hence

$$n(s, t, [u]) = \begin{cases} +1 & \text{if } s = t \leqslant v/2 \text{ and } v - 2s - u = \text{even}, \\ 0 & \text{otherwise}. \end{cases} \tag{38}$$

In form this is exactly the same branching rule as the one established in formula (20) for B_3. Here the branching reads

$$[v, 0, \ldots, 0] \to \sum_{\mu=0}^{v} \sum_{\nu=0}^{[(v-\mu)/2]} \left(\frac{\mu}{2}, \frac{\mu}{2}, [v-\mu-2\nu, 0, \ldots, 0]\right)' \tag{39}$$

and has the property that again no degeneracy occurs.

Although this completes our proof of the branching rule, which is only dependent on the rank n of the algebra B_n through the number of zeros between the square brackets, on the left- and right-hand sides of (39), we want to add two comments. Firstly, it is worthwhile to investigate as a verification of the validity of (39) whether the dimension of the B_n representation $[v, 0, \ldots, 0]$ is equal to the sum of the dimensions of the representations in which $[v, 0, \ldots, 0]$ reduces according to (39). This calculation is executed in the appendix. Secondly, it has to be noticed that (39) can be repetitively used together with (21) in order to obtain the branching rule for the reduction of symmetric representations of B_n $(n > 3)$ into representations of the maximal direct sum of A_1 subalgebras of B_n, i.e. $B_n \to A_1^1 \oplus A_1^1 \oplus \ldots \oplus A_1^1 \oplus A_1^\alpha$ (n terms), where $\alpha = 1$ if n = even and $\alpha = 2$ if n = odd. It is readily seen that in this reduction degeneracies arise.

4. Representation of the generators

The algebra B_n is an $n(2n+1)$-dimensional vector space spanned by $n(2n+1)$ basic generators (infinitesimal operators), which can be freely chosen. In many theoretical investigations the Cartan–Weyl basis is a common choice. However, when considering the reduction of an algebra into a regular subalgebra it is useful to construct a basis of which part of the basic generators span the subalgebra. The question is whether the remaining generators can be constructed such that they form a representation of the subalgebra. We want to state here, without explicit construction, that this is indeed the case for the algebra–subalgebra system $B_n \supset B_{n-2} \oplus A_1 \oplus A_1$.

Clearly the dimension of the regular subalgebra is $(n-2)(2n-3)+3+3 = 2n^2 - 7n + 12$. Consequently, the dimension of the vector space which is the complement of $B_{n-2} \oplus A_1 \oplus A_1$ in B_n is $4(2n-3)$. An evident factorisation of this number, keeping in mind the obvious symmetry with respect to both A_1's, is $(2n-3) \times 2 \times 2$. Now $(2n-3)$ is precisely the dimension of the vector representation $[1, 0, \ldots, 0]$ of B_{n-2}, as may be verified from the dimension formula (A1). On the other hand, 2 is the dimension of the spinor representation $[\frac{1}{2}]$ of A_1. Hence, we state that in the complementary vector space it is possible to assign a basis of generators which are the elements of a mixed tensor–spinor–spinor representation $T^{[1,0,\ldots,0],1/2,1/2}$ of $B_{n-2} \oplus A_1 \oplus A_1$. A rigorous proof of this statement is based on the observation that the generators of B_n themselves form the representation $[1, 1, 0, \ldots, 0]$ of B_n. Now, the branching of this particular (non-symmetric) representation can be derived by means of the method of § 3. We shall not insist on the details of the calculation, and merely state the result

$$[1,1,0,\ldots,0] \to [(1,1,\ldots,0],0,0)'+([0,\ldots,0],1,0)'+([0,\ldots,0],0,1)' + ([1,0,\ldots,0],\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_n \to \mathrm{B}_{n-2}\oplus\mathrm{A}_1\oplus\mathrm{A}_1) \qquad (n\geqslant 4) \tag{40}$$

which clearly confirms the intuitive hypothesis. If, moreover, we take into consideration that

$$[1,1,0]\to(1,0,0)'+(0,1,0)'+(0,0,1)'+(1,\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_3\to\mathrm{A}_1\oplus\mathrm{A}_1\oplus\mathrm{A}_1),$$

$$[1,1]\to(1,0)'+(0,1)'+(\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_2\to\mathrm{A}_1\oplus\mathrm{A}_1),$$

$$[1,0,\ldots,0]\to([1,0,\ldots,0],0,0)'+([0,\ldots,0],\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_n\to\mathrm{B}_{n-2}\oplus\mathrm{A}_1\oplus\mathrm{A}_1) \qquad (n\geqslant 4), \tag{41}$$

$$[1,0,0]\to(1,0,0)'+(0,\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_3\to\mathrm{A}_1\oplus\mathrm{A}_1\oplus\mathrm{A}_1),$$

$$[1,0]\to(0,0)'+(\tfrac{1}{2},\tfrac{1}{2})' \qquad (\mathrm{B}_2\to\mathrm{A}_1\oplus\mathrm{A}_1),$$

we can iteratively apply (40) and (41) in order to find out to what representation of $\mathrm{A}_1\oplus\mathrm{A}_1\oplus\ldots\oplus\mathrm{A}_1$ (n times) the generators of B_n which are not generators of the n A_1 subalgebras belong. It turns out that a distinction must be made between even and odd n values. In the first case all these generators behave as spinor components, in the latter case some of the generators belong to the vector representation of the subalgebra A_1^2. In any case, however, generators do not arrange in one mixed tensor–spinor operator, but in a sum of these.

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Appendix

According to Judd (1963), we have for the dimension of the irreducible representation $[w_1, w_2, \ldots, w_n]$ of B_n the formula

$$D_n[w_1,w_2,\ldots,w_n]=\prod_{\alpha^+}\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{\Lambda}+\boldsymbol{R})}{\boldsymbol{\alpha}\cdot\boldsymbol{R}}$$

where the product extends over all positive roots of B_n.

Hence, for the symmetric representations we obtain

$$\begin{aligned} D_n[v,0,\ldots,0] &= \frac{v+n-\frac{1}{2}}{n-\frac{1}{2}}\prod_{j=2}^{n}\frac{(v+n-\frac{1}{2})^2-(n-j+\frac{1}{2})^2}{(n-\frac{1}{2})^2-(n-j+\frac{1}{2})^2} \\ &= \frac{2v+2n-1}{2n-1}\prod_{j=2}^{n}\frac{(v+2n-j)(v+j-1)}{(2n-j)(j-1)} \\ &= \frac{2v+2n-1}{2n-1}\binom{v+2n-2}{2n-2}=\binom{v+2n-1}{2n-1}+\binom{v+2n-2}{2n-1}. \end{aligned} \tag{A1}$$

As a consequence of the branching rule (39) we should be able to prove that

$$D_{n+2}[v,0,\ldots,0]=\sum_{s=0}^{v/2}{}' (2s+1)^2 \sum_{u=0,1}^{v-2s}{}'' D_n[u,0,\ldots,0] \tag{A2}$$

where Σ' signifies that the summation index increases in steps of $\frac{1}{2}$ and Σ'' that it increases by 2. In the latter case the summation index starts at 0 if $v-2s$ is even and at 1 if it is odd. (A2) can be transformed into

$$\begin{aligned}
D_{n+2}[v,0,\ldots,0] &= \sum_{s=0}^{v} (v+1-s)^2 \sum_{u=0,1}^{s}{}'' D_n[u,0,\ldots,0]\\
&= \sum_{s=0}^{v} (v+1-s)^2 \sum_{u=0,1}^{s}{}'' \left[\binom{u+2n-1}{2n-1}+\binom{u+2n-2}{2n-1}\right]\\
&= \sum_{s=0}^{v} (v+1-s)^2 \sum_{u=0}^{s} \binom{u+2n-1}{2n-1}\\
&= \sum_{s=0}^{v} (v+1-s)^2 \binom{2n+s}{2n}.
\end{aligned}$$

A well known summation formula for binomial coefficients has been used here (Gradshteyn and Ryzhik 1965). On the use of the properties

$$\sum_{k=0}^{m} k\binom{n+k}{n}=(n+1)\binom{n+m+1}{n+2},$$

$$\sum_{k=0}^{m} k^2\binom{n+k}{n}=(n+1)(n+2)\binom{n+m+1}{n+3}+(n+1)\binom{n+m+1}{n+2},$$

we are able to verify the branching rule. Indeed,

$$\begin{aligned}
&\sum_{s=0}^{v} (v+1-s)^2\binom{2n+s}{2n}\\
&\quad=(v+1)^2\sum_{s=0}^{v}\binom{2n+s}{2n}-2(v+1)\sum_{s=0}^{v} s\binom{2n+s}{2n}+\sum_{s=0}^{v} s^2\binom{2n+s}{2n}\\
&\quad=(2n+1)(2n+2)\binom{v+2n+1}{2n+3}-(2v+1)(2n+1)\binom{v+2n+1}{2n+2}\\
&\qquad+(v+1)^2\binom{v+2n+1}{2n+1}\\
&\quad=\frac{2v+2n+3}{2n+3}\binom{v+2n+2}{2n+2}\\
&\qquad\times\frac{(v^2+2v+1)(4n^2+10n+6)-(2nv+v)(4v+2nv+4n+5)}{(2v+2n+3)(v+2n+2)}
\end{aligned}$$

It is easy to verify that this expression reduces to the one for $D_{n+2}\,[v,0,\ldots,0]$ as given by (A1).

Note added in proof. We thank Dr R C King (Southampton) for communicating to us that the branching rule (39) can be derived as a special case of much more general branching rule theorems (King 1975, King *et al* 1981, Wybourne 1970).

References

Dynkin E B 1965a *Am. Math. Soc. Transl.* **6** 111
—— 1965b *Am. Math. Soc. Transl.* **6** 245
Gradshteyn I S and Ryzhik I M 1965 *Table of integrals, series and products* (New York: Academic) p 3
Judd B R 1963 *Operator techniques in atomic spectroscopy* (New York: McGraw-Hill)
Kemmer N, Pursey D L and Williams S A 1968 *J. Math. Phys.* **9** 1224–30
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509
McKay W G and Patera J 1981 *Table of dimensions, indices and branching rules for representations of simple Lie algebras* (New York: Marcel Dekker)
Stone A P 1970 *J. Math. Phys.* **11** 29–38
Williams S A and Pursey D L 1968 *J. Math. Phys.* **9** 1230–40
Wybourne B G 1970 *Symmetry principles and atomic spectroscopy* (New York: Wiley Interscience)
—— 1974 *Classical groups for physicists* (New York: Wiley Interscience)